2/5/2018

$F : [\alpha, b] \to \mathbb{R}$

$$F(x) = \begin{cases} x^2 \sin(1/x^2), & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

F παραγ. στο $[0,1]$

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$\exists\, G'_+(0) = \lim_{x\to 0^+} \dfrac{F(x) - F(0)}{x - 0} = \lim_{x\to 0^+} \dfrac{1}{x}\left[x^2 \sin(1/x^2)\right] = 0$

$$F(x) = 2x \underbrace{\sin x\left(\frac{1}{x^2}\right)}_{x\to 0^+} - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) \quad x \in [0,1]$$

$F'(0) = F'_+(0) = 0 \qquad x_n \to 0^+,\; x_n = \dfrac{1}{\sqrt{2\kappa n}} \quad \dfrac{1}{x_n^2} = 2\kappa n$

$f'(x_n) \to -\infty$

$F'(x_n) \underset{n\to\infty}{\cong} -\dfrac{2}{x^4} < \to -\infty$

Θεώρημα: $G\,[a,b] \to \mathbb{R}$ παράγ.

G': R-ολοκλ. $= \int_a^b G' = G(b) - G(a)$

Αποδ: $P = \{a = x_0 < x_1 < \dots < x_{n-1} < x_n = b\}$ τυχ. διαμ.

$L(G', P) = \sum_{k=0}^{n-1} m_k (x_{k+1} - x_k)$

$U(G', P) = \sum_{k=0}^{n-1} M_k (x_{k+1} - x_k)$

$m_k = \inf\{G'(x) : x \in [x_k, x_{k+1}]\}$
$M_k = \sup\{\quad -{\shortparallel}- \quad -{\shortparallel}- \quad\}$

$G/[x_k, x_{k+1}]$ Εφαρμ. ΘΜΤ $\Rightarrow$ $\exists\, \xi_k \in (x_k, x_{k+1})$:
$G'(\xi_k) = \dfrac{G(x_{k+1}) - G(x_k)}{x_{k+1} - x_k}$

$m_k \le G'(\xi_k) \le M_k \Rightarrow$

$$(x_{k+1} - x_k)\, m_k \le G(x_{k+1}) - G(x_k) \le M_k\, (x_{k+1} - x_k)$$

$$L(G', P) \le \sum_{k=0}^{n-1} [G(x_{k+1}) - G(x_k)] \le U(G', P) =$$

$$G(b) - G(a)$$

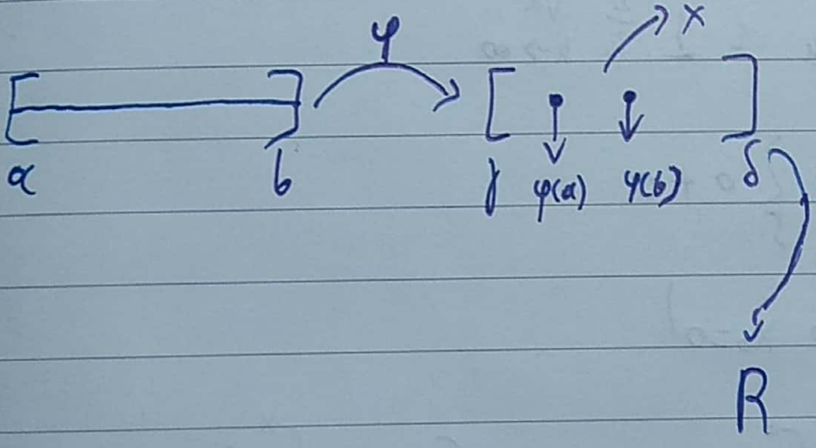
$$\int_a^b G' \le G(b) - G(a) \le \inf U(G', P) = \int_a^b G'$$

Θεώρημα: (αλλ. μεταβλ.)

Δίνεται $\varphi : [a,b] \to \mathbb{R}$: παραγωγίσιμη. φ' ολοκλ.

$$\left.\begin{array}{l} [a,b] \xrightarrow{\varphi} [\gamma, \delta] = I \\ \qquad\qquad \downarrow f \text{ συνx} \\ \quad \xrightarrow{F\circ\varphi} \qquad \mathbb{R} \end{array}\right\} \Rightarrow \int_a^b f(\varphi(t))\, \varphi'(t)\, dt$$

$$\int_{\varphi(a)}^{\varphi(b)} f(s)\, ds$$



$$F(x) = \int_\gamma^x f(s)\, ds$$

$f, F \Rightarrow F' = f \; (F'(s) = f(s) \quad \forall s \in [\gamma, \delta]$

$\searrow$ συνx

$F'(\varphi(t)) = f(\varphi(t)) \Leftarrow s = \varphi(t)$

$\forall t \in [a,b] \qquad t \in [a,b]$

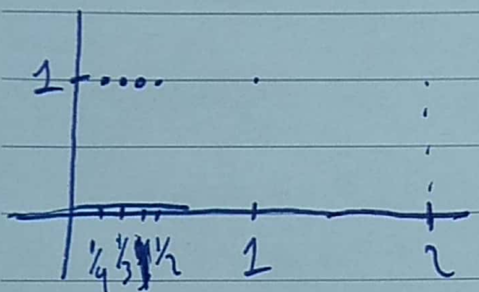
$t_1 = \int_a^b F'(\varphi(t))\,\varphi'(t)\,dt = \int_a^b (F\circ\varphi)'(t)\,dt = (F\circ\varphi)(b) - (F\circ\varphi)(a)$

$= \int_\gamma^{\varphi(b)} F - \int_\gamma^{\varphi(a)} F = \int_{\varphi(a)}^{\varphi(b)} F(s)\,ds$

$(F\circ\varphi)'(t) = [F(\varphi(t))]' = F'(\varphi(t))\,\varphi'(t)$

Ασκ. $F:[0,2]\to\mathbb{R}$ από τον τύπο

$F(x) = 1 \quad x\in\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\} \quad = 0$



$\int_0^2 F = ;$

Έστω $t>0$, μικρό θα δ.ο. $F/[t,2]$ ολοκλ.

$0 \le \int_1^2 F = \int_1^{1+t} F + \int_{1+t}^2 F = \int_1^{1+t} F + 0 < 1\,(1+t-1) = t \to 0^+$

$\int_0^2 F = \int_0^{1/n} F + \int_{1/n}^2 F = I_n + J_n \quad \le 1\,\frac{1}{n} \quad \forall n,\ n\to\infty$

$J_n = \int_{1/n}^2 F = \sum_{j=1}^{n-1} \int_{1/(j+1)}^{1/j} F + \int_1^2 F = \sum_j 0 + 0$

$F:[0,+\infty)\to\mathbb{R}$ (συνχ στο $x_0=0$)

Δ.ο. $\lim\limits_{x\to 0^+}^{\text{συνχ.}} \frac{1}{x} \int_0^x f(t)\,dt = f(0)$ $\qquad F(x) = \int_0^x f(t)$

$\left|\frac{1}{x}\int_0^x f(t)\,dt - f(0)\right| = \left|\frac{1}{x}\int_0^x f(t)\,dt - \frac{1}{x}\int_0^x f(0)\,dt\right| =$

$\frac{1}{x}\left|\int_0^x [f(t)-f(0)]dt\right|$ f συν. στο 0 Σταθ. $\epsilon>0 \Rightarrow \exists\delta>0$

$\forall t\geq 0,\ t<\delta \quad \forall t\in(0,\delta)$

$|f(t)-f(0)|<\epsilon$

$\lim_{x\to 0^+} I_x=0 \iff \forall\epsilon>0\ \exists\delta>0 . |I_x|<\epsilon\ \forall x\in(0,\delta]$

$$I_x=\frac{1}{x}\left|\int_0^x (f(t)-f(0))dt\right| \leq \frac{1}{x}\int_0^x |f(t)-f(0)|dt \leq$$

$x<\delta$ $\forall t\in(0,x]$

$$\frac{1}{x}\int_0^x \epsilon\, dt=\frac{1}{x}\epsilon x=\epsilon$$

Απ. $f,g:(a,b]\to\mathbb{R}$ $\mathbb{R}$-ολοκλ.

$$\Rightarrow\left(\int_a^b fg\right)^2\leq\left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$$

$\Pi(p)=\int_a^b(f+pg)^2dt\geq 0$ $p\in\mathbb{R}$ $\forall t\in[a,b]:[f(t)+pg(t)]^2\geq 0$

f,g ολοκλ. $\to pg$ ολοκλ. $(f+pg)^2$

$$\Pi(p)=\int_a^b [f^2(t)+2pf(t)g(t)+p^2g^2(t)]dt$$

$$=\left[\int_a^b f^2\right]+2p\left[\int_a^b f(t)g(t)\,dt\right]+p^2\int_a^b g^2\geq 0 \quad \forall p\in\mathbb{R}$$

$$\Delta=4\left(\int_a^b fg\,dt\right)^2-4\int_a^b f^2\int_a^b g^2\leq 0$$

Άσκ) $F:[\alpha,b]\to\mathbb{R}$
συνεχής
ώστε $\int_\alpha^b fg=0 \quad \forall g$ συνεχής $\Delta.0. \quad f=0$

Διαλέγουμε $g=f \Rightarrow \int_\alpha^b f^2=0$

$f^2(t)=h(t)\geq 0 \qquad \int_\alpha^b h=0 \Rightarrow h(x)=0 \quad \forall x$

$f^2(x)=0$

$f(x)=0 \quad \forall x$